Exercise 17

In Exercises 17–24, find the unknown if the solution of each equation is given:

$$\text{If } u(x) = e^{4x} \text{ is a solution of } u(x) = f(x) + 16\int_0^x (x-t)u(t)\,dt, \text{ find } f(x)$$

Solution

Substitute the solution into both sides of the equation.

$$e^{4x} = f(x) + 16\int_0^x (x-t)e^{4t}\,dt$$

Solve the integral with integration by parts. Let

$$\begin{aligned} v &= x - t & dw &= e^{4t}\,dt \\ dv &= -dt & w &= \frac{1}{4}e^{4t} \end{aligned}$$

and use the formula $\int v\,dw = vw - \int w\,dv$.

$$\begin{aligned} e^{4x} &= f(x) + 16\left[\frac{x-t}{4}e^{4t}\bigg|_0^x - \int_0^x \frac{1}{4}e^{4t}(-dt)\right] \\ &= f(x) + 16\left(-\frac{x}{4} + \frac{1}{4}\int_0^x e^{4t}\,dt\right) \\ &= f(x) + 16\left(-\frac{x}{4} + \frac{1}{16}e^{4t}\bigg|_0^x\right) \\ &= f(x) - 4x + e^{4x} - 1 \end{aligned}$$

Therefore,

$$f(x) = 4x + 1.$$